

NUMERICAL SOLUTION OF LINEAR INTEGRAL EQUATIONS USING MODIFIED BLOCK PULSE FUNCTIONS

Ayyubi Ahmad

Department of Mathematics, Faculty of Applied Science and Technology, Ahmad Dahlan University,
Jl. Ringroad Selatan, Kragilan, Tamanan, Banguntapan, Bantul, Yogyakarta, Indonesia
Email korespondensi: ayyubiahmad1@gmail.com

ABSTRACT

A computational method based on modified block pulse functions is proposed for solving numerically the linear Volterra and Fredholm integral equations. We obtain integration operational matrix of modified block pulse functions on interval $[0,1)$. A modified block pulse functions and their integration operational matrix can be reduced to a linear upper triangular system. Then, the problem under study is transformed to a system of linear algebraic equations which can be used to obtain an approximate solution of linear Volterra and Fredholm integral equations. Furthermore, the rate of convergence is $O(h)$ and error analysis of the proposed method are investigated. The results show that the approximate solutions have a good of efficiency and accuracy.

Keywords: Brownian Motion, Integration Operational Matrix, Itô Integral, Linear Integral Equations, ϵ MBPFs

1. Introduction

Linear integral equations and their solutions are great importance in the fields of science and engineering. Most physical problems, such as biological applications in population dynamics and genetics, where impulses occur naturally or are caused by controls, can be modeled by an integral equation. The linear Volterra and Fredholm integral equations each appear in the following form, respectively

$$f(x) = g(x) + \int_0^x k(x,y)f(y) dy, \quad x \in I = [0,1), \quad (1)$$

and

$$f(x) = g(x) + \int_0^1 k(x,y)f(y) dy, \quad x \in I, \quad (2)$$

where $f(x)$ is an unknown function, $g(x)$ and $k(x,y)$ are analytical function on I and $I \times I$, respectively.

Approximation theory is how functions can be approximated with simpler functions called base functions and with quantitatively characterizing the errors. One of these base functions is Block Pulse Functions (BPFs). BPFs are very common in use

and it seems their convergence is weak (Maleknejad and Rahimi, 2011). In fact by referring to error bound of BPFs approximation for achieving double precision, number of BPFs have to be doubled which means solving systems of equations with double unknowns and double equations (Maleknejad and Sohrabi, 2010).

In this study, the ϵ MBPFs were introduced and several theorems proved for the ϵ MBPFs method which showed that the results of this numerical extension were more accurate than the results of the block pulse extension. These functions are disjoint, orthogonal, and complete. According to the disjointness of ϵ MBPFs, the compound form will disappear at each subinterval when multiplication, division, and some other operations are applied. The orthogonality of ϵ MBPFs will cause the operational matrix to become a sparse matrix. The completeness of the ϵ MBPFs guarantees that an arbitrary small mean square error can be obtained for a finite real function having only a finite number of discontinuous points in the interval $x \in [0,1)$ by increasing the number of patterns in the modified block pulse series.

We use ε MBPFs and directly solve linear Volterra and Fredholm integral equations and by some examples to show the efficiency of ε MBPFs.

2. Materials and Methods

A set of ε modified block pulse functions $\psi_i(t), i = 0, 1, \dots, m$ on the interval $[0, T]$ are defined as

$$\psi_0(t) = \begin{cases} 1 & t \in [0, h - \varepsilon) = I_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_i(t) = \begin{cases} 1 & t \in [ih - \varepsilon, (i+1)h - \varepsilon) = I_i, \\ 0 & \text{yang lain,} \end{cases}$$

for $i = 1, 2, \dots, m-1$, and

$$\psi_m(t) = \begin{cases} 1 & t \in [T - \varepsilon, T) = I_m, \\ 0 & \text{otherwise.} \end{cases}$$

with a positive integer value for m and $h = \frac{T}{m}$.

Similar to BPFs, the important properties of ε MBPFs are as follows

- Disjointness:

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$$

where $i, j = 0, \dots, m$.

- Orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t) dt = h\delta_{ij},$$

where $i, j = 1, \dots, m-1$ and δ_{ij} is Kronecker delta.

- Completeness:

$$\int_0^T f^2(t) dt = \sum_{i=0}^{\infty} f_i^2 \|\psi_i(t)\|^2,$$

where

$$f_i = \frac{1}{\Delta(I_i)} \int_0^T f(t)\psi_i(t) dt, \quad (3)$$

and $\Delta(I_i)$ is length of interval I_i .

Rewriting Eq. (3) in the vector form we have

$$f(t) \simeq \sum_{i=0}^m f_i \psi_i(t) = F^T \Psi(t) = \Psi^T(t) F,$$

in which

$$F = (f_0 \ f_1 \ \dots \ f_m)^T,$$

and

$$\Psi(t) = (\psi_0(t) \ \psi_1(t) \ \dots \ \psi_m(t))^T. \quad (4)$$

Moreover, any two dimensional function $k(s, t) \in L^2([0, T_1] \times [0, T_2])$ can be expanded with respect to ε MBPFs such as

$$k(s, t) = \Psi^T(s) K \Psi(t) = \Psi^T(t) K^T \Psi(s),$$

where $\Psi(s)$ and $\Psi(t)$ are m_1 and m_2 dimensional ε MBPFs vectors respectively, and $K = (k_{ij}), i = 0, 1, \dots, m_1, j = 0, 1, \dots, m_2$ is the $m_1 \times m_2$ ε modified block pulse coefficient matrix with

$$k_{ij} = \frac{1}{\Delta(I_i)\Delta(I_j)} \int_0^{T_1} \int_0^{T_2} k(s, t) \Psi_i(s) \Psi_j(t) dt ds,$$

For convenience, we put $m_1 = m_2 = m$.

With defining $\Psi_{m+1}(t) = (\psi_0(t) \ \psi_1(t) \ \dots \ \psi_m(t))^T$, we have

$$\Psi_{m+1}(t) \Psi_{m+1}^T(t) = \begin{pmatrix} \psi_0(t) & 0 & \dots & 0 \\ 0 & \psi_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_m(t) \end{pmatrix}_{(m+1) \times (m+1)}.$$

Furthermore,

$$\Psi_{m+1}^T(t) \Psi_{m+1}(t) = 1,$$

and

$$\Psi_{m+1}(t) \Psi_{m+1}^T(t) F = D_F \Psi_{m+1}(t),$$

where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_0 \ f_1 \ \dots \ f_m)^T$.

Similar to BPFs,

$$\int_0^t \Psi_{m+1}(s) ds \simeq Q \Psi_{m+1}(t),$$

where the integration operational matrix Q of ε MBPFs is given by

$$Q = \begin{pmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & \dots & h-\varepsilon \\ 0 & \frac{h}{2} & \dots & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\varepsilon}{2} \end{pmatrix}_{(m+1) \times (m+1)}.$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s) ds \simeq \int_0^t F^T \Psi_{m+1}(s) ds \simeq F^T Q \Psi_{m+1}(t).$$

3. Results and Discussion

3.1 Volterra Integral Equations

We approximate functions $f(t)$, $g(t)$, and $k(s, t)$ in Eq. (1) with ε MBPFs as follows

$$\begin{aligned} f(t) &\simeq F^T \Psi(t), \\ g(t) &\simeq G^T \Psi(t), \end{aligned} \quad (5)$$

$$k(s, t) \simeq \Psi^T(s) K \Psi(t),$$

$\Psi(t)$ is defined by (4), F , G , and K are ε MBPFs coefficients of $f(t)$, $g(t)$, and $k(s, t)$, respectively. By substituting Eq. (5) in Eq. (1), we get

$$F^T \Psi(t) = G^T \Psi(t) + F^T \left(\int_0^t \Psi(s) \Psi^T(s) ds \right) K \Psi(t). \quad (6)$$

Let K^i be the i th row of the constant matrix K and R^i be the i th row of the integration operational matrix Q , D_{K^i} be a diagonal matrix with K^i as its diagonal entries. Assuming $m_1 = m_2$, we have

$$\begin{aligned} & \left(\int_0^t \Psi(s) \Psi^T(s) ds \right) K \Psi(t) \\ &= \begin{bmatrix} R^0 \Psi(t) & 0 & \dots & 0 \\ 0 & R^1 \Psi(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R^m \Psi(t) \end{bmatrix} \begin{bmatrix} K^0 \\ K^1 \\ \vdots \\ K^m \end{bmatrix} \Psi(t) \\ &= \begin{bmatrix} R^0 \Psi(t) K^0 \Psi(t) \\ R^1 \Psi(t) K^1 \Psi(t) \\ \vdots \\ R^m \Psi(t) K^m \Psi(t) \end{bmatrix} \\ &= \begin{bmatrix} R^0 \Psi(t) \Psi^T(t) K^{0T} \\ R^1 \Psi(t) \Psi^T(t) K^{1T} \\ \vdots \\ R^m \Psi(t) \Psi^T(t) K^{mT} \end{bmatrix} \\ &= \begin{bmatrix} R^0 D_{K^0} \\ R^1 D_{K^1} \\ \vdots \\ R^m D_{K^m} \end{bmatrix} \Psi(t) \\ &= B \Psi(t), \end{aligned}$$

where

$$B = \begin{bmatrix} k_{00} \left(\frac{h-\varepsilon}{2} \right) & 2k_{01}(h-\varepsilon) & \dots & 2k_{0m}(h-\varepsilon) \\ 0 & k_{11} \left(\frac{h}{2} \right) & \dots & 2k_{1m}(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{mm} \left(\frac{\varepsilon}{2} \right) \end{bmatrix}_{(m+1) \times (m+1)} \quad (7)$$

With substituting relations (7) and (8) in (6)

$$F^T \Psi(t) \simeq G^T \Psi(t) + F^T B \Psi(t).$$

Then,

$$F^T (I - B) \simeq G^T.$$

So, by setting $W = (I - B)^T$ and replacing $\simeq$ by $=$, we will have

$$WF = G.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficients of the unknown $f(t)$.

3.2 Fredholm Integral Equations

We approximate functions $f(x)$, $g(x)$, and $k(x, y)$ in Eq. (2) with ε MBPFs as follows

$$\begin{aligned} f(x) &\simeq F^T \Psi(t), \\ g(t) &\simeq G^T \Psi(t), \\ k(s, t) &\simeq \Psi^T(s) K \Psi(t), \end{aligned} \quad (9)$$

$\Psi(t)$ is defined by (4), F , G , and K are ε MBPFs coefficients of $f(t)$, $g(t)$, and $k(s, t)$, respectively. By substituting Eq. (9) in Eq. (2), we get

$$F^T \Psi(t) = G^T \Psi(t) + F^T \left(\int_0^1 \Psi(s) \Psi^T(s) ds \right) K \Psi(t), \quad (10)$$

Assuming $m_1 = m_2$ and using the relation $\int_0^1 \Psi(s) \Psi^T(s) ds = hI$ we have

$$\begin{aligned} \left(\int_0^1 \Psi(s) \Psi^T(s) ds \right) K \Psi(t) &= hI K \Psi(t) \\ &= T \Psi(t), \end{aligned} \quad (11)$$

where $T = hK$. With substituting (11) in (10)

$$F^T \Psi(t) \simeq G^T \Psi(t) + F^T T \Psi(t).$$

Then,

$$F^T (I - T) \simeq G^T.$$

So, by setting $Z = (I - T)^T$ and replacing $\simeq$ by $=$, we will have

$$ZF = G.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficients of the unknown $f(t)$.

3.3 Error Analysis

In the following theorems, for simplicity we assume $T = 1$ and $h = \frac{1}{m}$.

Theorem 1. If $\hat{f}_m(t) = \sum_{i=0}^m f_i \psi_i(t)$ and $f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t) \psi_i(t) dt$, $i = 0, \dots, m$ then:

- $\delta = \int_0^1 (f(t) - \sum_{i=0}^m f_i \psi_i(t))^2 dt$, achieves its minimum value.
- $\{\hat{f}_m(t)\}$ approaches $f(t)$ pointwise.
- $\int_0^1 f^2(t) dt = \sum_{i=0}^\infty f_i^2 \|\psi_i\|^2$.

[Proof]. Proof is like similar theorem in (Jiang, 1992) but intervals of integration have to redefine as I_i , $i = 0, \dots, m$ in (3.1). ■

Theorem 2. Assume:

- $f(t)$ is continuous and differentiable in $[-h, 1+h]$ with bounded derivative, that $|f'(t)| < M$.
- $\hat{f}_{ih}(t), i = 0, \dots, k-1$, are correspondingly BPFs, $\frac{h}{k}$ MBPFs ... , $\frac{(k-1)h}{k}$ MBPFs expansions of $f(t)$ base on $m+1$ ε MBPFs over interval $[0,1]$.
- $\bar{f}(t) = \frac{1}{k} \sum_{i=0}^{k-1} \hat{f}_{ih}(t)$.

Then

$$\|f(t) - \hat{f}_{ih}(t)\| = \mathcal{O}(h),$$

and

$$\|f(t) - \bar{f}(t)\| = \mathcal{O}\left(\frac{h}{k}\right) \text{ in } [h, 1-h].$$

[Proof]. Trapezoidal rule for integral is

$$\begin{aligned} \int_a^b f(t) dt &= \frac{b-a}{2} (f(a) + f(b)) \\ &\quad - \frac{(b-a)^3 f''(\eta)}{12} \\ &= \frac{b-a}{2} (f(a) + f(b)) + E, \end{aligned} \quad (12)$$

where $\eta \in [a, b]$ and E is error of integration.

Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_{i-1}, t_i]$. The representation error when $f(t)$ is represented by a series of BPFs over every subinterval $[t_i, t_i + \frac{h}{k}]$, $i = 0, \dots, m-1$ is

$$e_i(t) = f(t) - f_i \psi_i(t) = f(t) - f_i,$$

where $f_i = \frac{1}{h} \int_{ih}^{(i+1)h} f(t) dt$. From (12),

$$f_i = \frac{1}{2} (f(t_i) + f(t_i + h)) + E.$$

It is obvious that if $f(t) = C$ (constant), then $e_i(t) = 0$. So, this error is computed for $f(t) = t$ in interval $[t_i, t_i + \frac{h}{k}]$, $i = 1, \dots, m-1$.

For this function $E = 0$, so

$$\begin{aligned} e_i(t)_{[t_i, t_i + \frac{h}{k}]} &= |t - f_i| \\ &= \left| t - \frac{t_i + t_{i+1}}{2} \right| \\ &= \left| t - \left(t_i + \frac{h}{2} \right) \right| \\ &\leq \frac{h}{2}. \end{aligned}$$

Then this error with BPFs is $\frac{h}{2} M$.

Similarly, the error when $f(t)$ is represented

in a series of ε MBPFs over every subinterval $[t_i, t_i + \frac{h}{k}]$ is

$$\begin{aligned} e_i(t)_{[t_i, t_i + \frac{h}{k}]} &= \left| t - \left(\frac{\sum_{j=0}^{k-1} \left(t_i - \left(\frac{jh}{k} \right) + t_{i+1} - \left(\frac{jh}{k} \right) \right)}{2k} \right) \right| \\ &= \left| t - \left(\frac{\sum_{j=0}^{k-1} \left(t_i - \left(\frac{jh}{k} \right) + t_i + h - \left(\frac{jh}{k} \right) \right)}{2k} \right) \right| \\ &= \left| t - \left(t_i + \frac{h}{2} \right) - \frac{(k-1)h}{2k} \right| \\ &\leq \frac{h}{2k}. \end{aligned}$$

So, the error with ε MBPFs is $\frac{h}{2k} M$.

For I_0 in $[0, \frac{h}{k}]$ we have

$$\begin{aligned} e_i(t)_{[0, \frac{h}{k}]} &= \left| t - \sum_{j=0}^{k-1} \frac{h - \left(\frac{jh}{k} \right)}{2k} \right| \\ &= \left| t - \left(\frac{h}{2} - \frac{(k-1)h}{4k} \right) \right| \\ &= \left| t - \left(\frac{h}{4} + \frac{h}{4k} \right) \right| \\ &= \mathcal{O}\left(\frac{h}{4}\right). \end{aligned}$$

So, the error is $\mathcal{O}\left(\frac{h}{4}\right)$ also for I_n .

Now,

$$\begin{aligned} \|e_i(t)\|^2 &= \int_{t_i}^{t_i + \frac{h}{k}} |e_i(t)|^2 dt \\ &= \int_{t_i}^{t_i + \frac{h}{k}} \frac{h^2}{4k^2} M^2 dt \\ &= \frac{h^3}{4k^3} M^2, \\ \|e\|^2 &= \int_0^1 e^2(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \left(\sum_{i=1}^m \sum_{j=0}^{k-1} e_i(t) \right)^2 dt \\
 &= \sum_{i=1}^m \sum_{j=0}^{k-1} \int_0^1 e_i^2(t) dt \\
 &= \sum_{i=1}^m \sum_{j=0}^{k-1} \|e_i(t)\|^2 \\
 &= \frac{1}{h} \cdot k \cdot \frac{h^3}{4k^3} M^2 \\
 &= \frac{h^2}{4k^2} M^2.
 \end{aligned}$$

We define the representation error between $f(s, t)$ and its 2D- ε MBPFs expansion f_{ij} over every subregion D_{ij} , is defined as

$$e_{ij}(s, t) = f(s, t) - f_{ij},$$

where $D_{ij} := \left\{ (s, t) \mid t_i \leq s \leq t_i + \frac{h}{k}, t_j \leq t \leq t_j + \frac{h}{k} \right\}$.

With Taylor's expansion and similarity to the above discussion,

$$\|e(s, t)\| = \frac{h}{2k} M. \quad \blacksquare$$

Theorem 3. Assume that

- $P(\omega \in \Omega : \|u(\omega, t)\| < C) = 1.$
- $\|k_i\| < C, i = 1, 2.$

Then

$$\sup(E(\|(u - \bar{u})\|^2))^{\frac{1}{2}} = O\left(\frac{h}{k}\right), \quad t \in [h, 1 - h].$$

$0 \leq t \leq T$

[Proof]. For a complete proof see (Maleknejad, 2014). ■

3.4 Numerical Examples

Example 1. Consider the following linear Volterra integral equation,

$$f(t) = 5t^3 - t^5 + \int_0^t sf(s) ds,$$

where $s, t \in [0, 1)$. The exact solution is $f(t) = 5t^3$.

The exact and approximation solution are shown in Table 1. The curves in Fig. 1 represents a trajectory of the exact and approximation solution computed by the presented method.

Table 1. The exact and approximate solution of Example 1 for $m = 65$

t	Exact	Approximation
0	0	0.000001112
0.1	0.005	0.00757
0.2	0.04	0.04811
0.3	0.135	0.1505
0.4	0.32	0.3433
0.5	0.625	0.6545
0.6	1.08	1.114
0.7	1.715	1.75
0.8	2.56	2.59
0.9	3.645	3.665
1	5	4.942

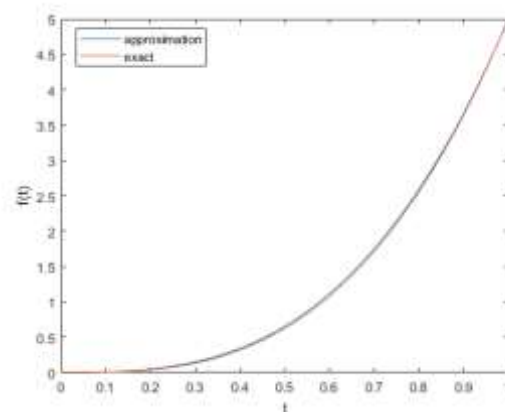


Figure 1. The trajectory of the exact and approximation solution of example 1.

Example 2. Consider the following linear Fredholm integral equation,

$$f(t) = t + e^t - \frac{4}{3} + \int_0^1 sf(s) ds,$$

where $t \in [0, 1)$. The exact solution is $f(t) = t + e^t$.

The exact and approximation solution are shown in Table 2. The curves in Fig. 2 represents a trajectory of the exact and approximation solution computed by the presented method.

Table 2. The exact and approximate solution of Example 2 for $m = 65$

t	Exact	Approximation
0	1	1.064
0.1	1.205171	1.291
0.2	1.421403	1.505
0.3	1.649859	1.732
0.4	1.891825	1.972
0.5	2.148721	2.226
0.6	2.422119	2.496
0.7	2.713753	2.784
0.8	3.025541	3.092
0.9	3.359603	3.422
1	3.718282	3.761

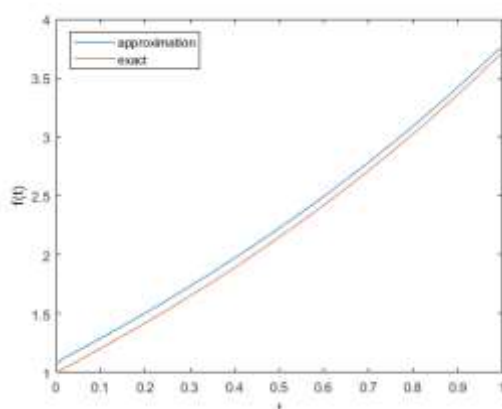


Figure 2. The trajectory of the exact and approximation solution of example 2.

4. Conclusions

The ε MBPFs and their integration operational matrix are used to obtain the solution of linear Volterra and Fredholm integral equations. The present method reduces a linear Volterra and Fredholm integral equations into a system of algebraic equations. The matrices Q have many zeros and the method is computationally very attractive. Its applicability and accuracy of the proposed method was checked on some examples. The results show that the approximate solutions of the proposed method have a good of efficiency and accuracy.

5. References

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